Modal Information Logics

Søren Brinck Knudstorp
Annual VvL Seminar, Utrecht University, Master's Thesis Award Presentation
Supervised by **Johan van Benthem** and **Nick Bezhanishvili**
December 08, 2023

University of Amsterdam

Plan for the talk

- Introducing the logics
- Stating the problems
- Outlining the strategy
- Solving the problems using the strategy

Defining (the basic) modal information logics (MILs)

Definition (language and semantics)

The language is given by

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle \mathrm{sup} \rangle \varphi\psi,$$

and the semantics of '$\langle \mathrm{sup} \rangle$' is:

$$w \Vdash \langle \mathrm{sup} \rangle \varphi\psi \quad \text{iff} \quad \exists u, v (u \Vdash \varphi;\ v \Vdash \psi;\ w = \sup\{u, v\})$$

Example

Definition (frames and logics)

Three classes of frames $(W, \leq)$, namely those where

- *(Pre)* $(W, \leq)$ is a preorder (refl., tr.);
- *(Pos)* $(W, \leq)$ is a poset (anti-sym. preorder); and
- *(Sem)* $(W, \leq)$ is a join-semilattice (poset w. all bin. joins)

Resulting in the logics MIL_{Pre}, MIL_{Pos}, MIL_{Sem}, respectively.

Defining (the basic) modal information logics (MILs)

Definition (language and semantics)

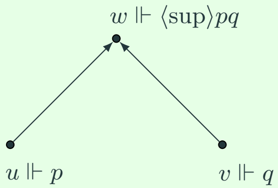
The language is given by

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle \mathrm{sup} \rangle \varphi\psi,$$

and the semantics of '$\langle \mathrm{sup} \rangle$' is:

$$w \Vdash \langle \mathrm{sup} \rangle \varphi\psi \quad \text{iff} \quad \exists u, v (u \Vdash \varphi;\ v \Vdash \psi; \\ w = \sup\{u, v\})$$

Example



Definition (frames and logics)

Three classes of frames $(W, \leq)$, namely those where

(Pre) $(W, \leq)$ is a preorder (refl., tr.);

(Pos) $(W, \leq)$ is a poset (anti-sym. preorder); and

(Sem) $(W, \leq)$ is a join-semilattice (poset w. all bin. joins)

Resulting in the logics MIL_{Pre}, MIL_{Pos}, MIL_{Sem}, respectively.

Defining (the basic) modal information logics (MILs)

Definition (language and semantics)

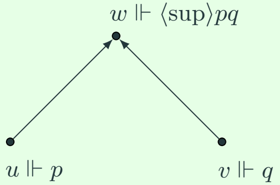
The language is given by

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle \sup \rangle \varphi\psi,$$

and the semantics of '$\langle \sup \rangle$' is:

$$w \Vdash \langle \sup \rangle \varphi\psi \quad \text{iff} \quad \exists u, v (u \Vdash \varphi;\ v \Vdash \psi;\ w = \sup\{u, v\})$$

Example



Definition (frames and logics)

Three classes of frames $(W, \leq)$, namely those where

- *(Pre)* $(W, \leq)$ is a preorder (refl., tr.);
- *(Pos)* $(W, \leq)$ is a poset (anti-sym. preorder); and
- *(Sem)* $(W, \leq)$ is a join-semilattice (poset w. all bin. joins)

Resulting in the logics MIL_{Pre}, MIL_{Pos}, MIL_{Sem}, respectively.

Motivation

Why MILs?

- Connect with other logics (e.g., truthmaker semantics).
- Introduced to model a theory of information (by van Benthem (1996)).
- Modestly extend $\mathbf{S4}$ [MIL_{Pre}, MIL_{Pos}].

What in particular?

Guided by two central problems (posed in van Benthem (2017, 2019)), namely

(A) axiomatizing MIL_{Pre} and MIL_{Pos}; and

(D) proving (un)decidability.

Motivation

Why MILs?

- Connect with other logics (e.g., truthmaker semantics).
- Introduced to model a theory of information (by van Benthem (1996)).
- Modestly extend $\mathbf{S4}$ $[MIL_{Pre}, MIL_{Pos}]$.

What in particular?

Guided by two central problems (posed in van Benthem (2017, 2019)), namely

(A) axiomatizing MIL_{Pre} and MIL_{Pos}; and

(D) proving (un)decidability.

Motivation

Why MILs?

- Connect with other logics (e.g., truthmaker semantics).
- Introduced to model a theory of information (by van Benthem (1996)).
- Modestly extend $\mathbf{S4}$ [MIL_{Pre}, MIL_{Pos}].

What in particular?

Guided by two central problems (posed in van Benthem (2017, 2019)), namely

(A) axiomatizing MIL_{Pre} and MIL_{Pos}; and

(D) proving (un)decidability.

Motivation

Why MILs?

- Connect with other logics (e.g., truthmaker semantics).
- Introduced to model a theory of information (by van Benthem (1996)).
- Modestly extend **S4** [MIL_{Pre}, MIL_{Pos}].

What in particular?

Guided by two central problems (posed in van Benthem (2017, 2019)), namely

(A) axiomatizing MIL_{Pre} and MIL_{Pos}; and

(D) proving (un)decidability.

Motivation

Why MILs?

- Connect with other logics (e.g., truthmaker semantics).
- Introduced to model a theory of information (by van Benthem (1996)).
- Modestly extend $\mathbf{S4}$ [MIL_{Pre}, MIL_{Pos}].

What in particular?

Guided by two central problems (posed in van Benthem (2017, 2019)), namely

(A) axiomatizing MIL_{Pre} and MIL_{Pos}; and

(D) proving (un)decidability.

Motivation

Why MILs?

- Connect with other logics (e.g., truthmaker semantics).
- Introduced to model a theory of information (by van Benthem (1996)).
- Modestly extend $\mathbf{S4}$ [MIL_{Pre}, MIL_{Pos}].

What in particular?

Guided by two central problems (posed in van Benthem (2017, 2019)), namely

(A) axiomatizing MIL_{Pre} and MIL_{Pos}; and
(D) proving (un)decidability.

Initial study (MIL_{Pre} and MIL_{Pos})

Proposition

MILs lack the finite model property (FMP) w.r.t. their classes of definition.

How we solve (A), and then (D) using (A):

(1) We axiomatize MIL_{Pre} (and deduce $MIL_{Pre} = MIL_{Pos}$).

(2) Use the axiomatization to find another class of structures $\mathcal{C}$ for which $\mathrm{Log}(\mathcal{C}) = MIL_{Pre}$.

(3) Prove that on $\mathcal{C}$ we do have the FMP and deduce decidability.

Initial study (MIL_{Pre} and MIL_{Pos})

Proposition

MILs lack the finite model property (FMP) w.r.t. their classes of definition.

How we solve (A), and then (D) using (A):

(1) We axiomatize MIL_{Pre} (and deduce $MIL_{Pre} = MIL_{Pos}$).

(2) Use the axiomatization to find another class of structures $\mathcal{C}$ for which $\mathrm{Log}(\mathcal{C}) = MIL_{Pre}$.

(3) Prove that on $\mathcal{C}$ we do have the FMP and deduce decidability.

Initial study (MIL_{Pre} and MIL_{Pos})

Proposition

MILs lack the finite model property (FMP) w.r.t. their classes of definition.

How we solve (A), and then (D) using (A):

(1) We axiomatize MIL_{Pre} (and deduce $MIL_{Pre} = MIL_{Pos}$).

(2) Use the axiomatization to find another class of structures $\mathcal{C}$ for which $\mathrm{Log}(\mathcal{C}) = MIL_{Pre}$.

(3) Prove that on $\mathcal{C}$ we do have the FMP and deduce decidability.

Initial study (MIL_{Pre} and MIL_{Pos})

Proposition

MILs lack the finite model property (FMP) w.r.t. their classes of definition.

How we solve (A), and then (D) using (A):

(1) We axiomatize MIL_{Pre} (and deduce $MIL_{Pre} = MIL_{Pos}$).

(2) Use the axiomatization to find another class of structures $\mathcal{C}$ for which $\mathrm{Log}(\mathcal{C}) = MIL_{Pre}$.

(3) Prove that on $\mathcal{C}$ we do have the FMP and deduce decidability.

Initial study (MIL_{Pre} and MIL_{Pos})

Proposition

MILs lack the finite model property (FMP) w.r.t. their classes of definition.

How we solve (A), and then (D) using (A):

(1) We axiomatize MIL_{Pre} (and deduce $MIL_{Pre} = MIL_{Pos}$).

(2) Use the axiomatization to find another class of structures $\mathcal{C}$ for which $\mathrm{Log}(\mathcal{C}) = MIL_{Pre}$.

(3) Prove that on $\mathcal{C}$ we do have the FMP and deduce decidability.

(1): axiomatizing MIL_{Pre}

Axiomatization (soundness and completeness)

MIL_{Pre} is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \to \langle \text{sup} \rangle pq$

(4) $\langle P \rangle \langle P \rangle p \to \langle P \rangle p$

(Co.) $\langle \text{sup} \rangle pq \to \langle \text{sup} \rangle qp$

(Dk.) $(p \wedge \langle \text{sup} \rangle qr) \to \langle \text{sup} \rangle pq$

Proof idea

Soundness ✓

For completeness, let $\Gamma \supseteq \Gamma_0$ be an MCS extending some consistent Γ_0. We construct a satisfying model using the step-by-step method:

(Base) Singleton frame $\mathbb{F}_0 := (\{x_0\}, \{(x_0, x_0)\})$ and 'labeling' $l_0(x_0) = \Gamma$.

(Ind) Suppose $(\mathbb{F}_n, l_n)$ has been constructed.
– If $x \in \mathbb{F}_n$ and $\neg\langle \sup \rangle \psi\psi' \in l_n(x)$ but $x = \sup_n\{y, z\}$ s.t. $\psi \in l_n(y), \psi' \in l_n(z)$, *coherently* extend to $(\mathbb{F}_{n+1}, l_{n+1}) \supseteq (\mathbb{F}_n, l_n)$ so that $x \neq \sup_{n+1}\{y, z\}$.
– Similarly, for $\langle \sup \rangle \chi\chi' \in l_n(x)$.

(1): axiomatizing MIL_{Pre}

Axiomatization (soundness and completeness)

MIL_{Pre} is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \rightarrow \langle \text{sup} \rangle pq$

(4) $\langle P \rangle \langle P \rangle p \rightarrow \langle P \rangle p$

(Co.) $\langle \text{sup} \rangle pq \rightarrow \langle \text{sup} \rangle qp$

(Dk.) $(p \wedge \langle \text{sup} \rangle qr) \rightarrow \langle \text{sup} \rangle pq$

Proof idea

Soundness ✓

For completeness, let $\Gamma \supseteq \Gamma_0$ be an MCS extending some consistent Γ_0. We construct a satisfying model using the step-by-step method:

(Base) Singleton frame $\mathbb{F}_0 := (\{x_0\}, \{(x_0, x_0)\})$ and 'labeling' $l_0(x_0) = \Gamma$.

(Ind) Suppose $(\mathbb{F}_n, l_n)$ has been constructed.

– If $x \in \mathbb{F}_n$ and $\neg\langle \text{sup} \rangle \psi\psi' \in l_n(x)$ but $x = \text{sup}_n\{y, z\}$ s.t. $\psi \in l_n(y), \psi' \in l_n(z)$, *coherently* extend to $(\mathbb{F}_{n+1}, l_{n+1}) \supseteq (\mathbb{F}_n, l_n)$ so that $x \neq \text{sup}_{n+1}\{y, z\}$.

– Similarly, for $\langle \text{sup} \rangle \chi\chi' \in l_n(x)$.

(1): axiomatizing MIL_{Pre}

Axiomatization (soundness and completeness)

MIL_{Pre} is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \to \langle \text{sup} \rangle pq$

(4) $\langle P \rangle \langle P \rangle p \to \langle P \rangle p$

(Co.) $\langle \text{sup} \rangle pq \to \langle \text{sup} \rangle qp$

(Dk.) $(p \wedge \langle \text{sup} \rangle qr) \to \langle \text{sup} \rangle pq$

Proof idea

Soundness ✓

For completeness, let $\Gamma \supseteq \Gamma_0$ be an MCS extending some consistent Γ_0. We construct a satisfying model using the step-by-step method:

(Base) Singleton frame $\mathbb{F}_0 := (\{x_0\}, \{(x_0, x_0)\})$ and 'labeling' $l_0(x_0) = \Gamma$.

(Ind) Suppose $(\mathbb{F}_n, l_n)$ has been constructed.
– If $x \in \mathbb{F}_n$ and $\neg\langle \text{sup} \rangle \psi\psi' \in l_n(x)$ but $x = \text{sup}_n\{y, z\}$ s.t. $\psi \in l_n(y), \psi' \in l_n(z)$, *coherently* extend to $(\mathbb{F}_{n+1}, l_{n+1}) \supseteq (\mathbb{F}_n, l_n)$ so that $x \neq \text{sup}_{n+1}\{y, z\}$.
– Similarly, for $\langle \text{sup} \rangle \chi\chi' \in l_n(x)$.

(1): axiomatizing MIL_{Pre}

Axiomatization (soundness and completeness)

MIL_{Pre} is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \rightarrow \langle \mathsf{sup} \rangle pq$

(4) $\langle P \rangle \langle P \rangle p \rightarrow \langle P \rangle p$

(Co.) $\langle \mathsf{sup} \rangle pq \rightarrow \langle \mathsf{sup} \rangle qp$

(Dk.) $(p \wedge \langle \mathsf{sup} \rangle qr) \rightarrow \langle \mathsf{sup} \rangle pq$

Proof idea

Soundness ✓

For completeness, let $\Gamma \supseteq \Gamma_0$ be an MCS extending some consistent Γ_0. We construct a satisfying model using the step-by-step method:

(Base) Singleton frame $\mathbb{F}_0 := (\{x_0\}, \{(x_0, x_0)\})$ and 'labeling' $l_0(x_0) = \Gamma$.

(Ind) Suppose $(\mathbb{F}_n, l_n)$ has been constructed.

– If $x \in \mathbb{F}_n$ and $\neg\langle \mathrm{sup} \rangle \psi\psi' \in l_n(x)$ but $x = \mathrm{sup}_n\{y, z\}$ s.t. $\psi \in l_n(y), \psi' \in l_n(z)$, *coherently* extend to $(\mathbb{F}_{n+1}, l_{n+1}) \supseteq (\mathbb{F}_n, l_n)$ so that $x \neq \mathrm{sup}_{n+1}\{y, z\}$.

– Similarly, for $\langle \mathrm{sup} \rangle \chi\chi' \in l_n(x)$.

(1): axiomatizing MIL_{Pre}

Axiomatization (soundness and completeness)

MIL_{Pre} is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \to \langle \text{sup} \rangle pq$

(4) $\langle P \rangle \langle P \rangle p \to \langle P \rangle p$

(Co.) $\langle \text{sup} \rangle pq \to \langle \text{sup} \rangle qp$

(Dk.) $(p \wedge \langle \text{sup} \rangle qr) \to \langle \text{sup} \rangle pq$

Proof idea

Soundness ✓

For completeness, let $\Gamma \supseteq \Gamma_0$ be an MCS extending some consistent Γ_0. We construct a satisfying model using the step-by-step method:

(Base) Singleton frame $\mathbb{F}_0 := (\{x_0\}, \{(x_0, x_0)\})$ and 'labeling' $l_0(x_0) = \Gamma$.

(Ind) Suppose $(\mathbb{F}_n, l_n)$ has been constructed.
– If $x \in \mathbb{F}_n$ and $\neg\langle \text{sup} \rangle \psi\psi' \in l_n(x)$ but $x = \text{sup}_n\{y, z\}$ s.t. $\psi \in l_n(y), \psi' \in l_n(z)$, *coherently* extend to $(\mathbb{F}_{n+1}, l_{n+1}) \supseteq (\mathbb{F}_n, l_n)$ so that $x \neq \text{sup}_{n+1}\{y, z\}$.
– Similarly, for $\langle \text{sup} \rangle \chi\chi' \in l_n(x)$.

(1): axiomatizing MIL_{Pre}

Axiomatization (soundness and completeness)

MIL_{Pre} is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \to \langle \text{sup} \rangle pq$

(4) $\langle P \rangle \langle P \rangle p \to \langle P \rangle p$

(Co.) $\langle \text{sup} \rangle pq \to \langle \text{sup} \rangle qp$

(Dk.) $(p \wedge \langle \text{sup} \rangle qr) \to \langle \text{sup} \rangle pq$

Proof idea

Soundness ✓

For completeness, let $\Gamma \supseteq \Gamma_0$ be an MCS extending some consistent Γ_0. We construct a satisfying model using the step-by-step method:

(Base) Singleton frame $\mathbb{F}_0 := (\{x_0\}, \{(x_0, x_0)\})$ and 'labeling' $l_0(x_0) = \Gamma$.

(Ind) Suppose $(\mathbb{F}_n, l_n)$ has been constructed.

– If $x \in \mathbb{F}_n$ and $\neg\langle \sup \rangle \psi\psi' \in l_n(x)$ but $x = \sup_n\{y, z\}$ s.t. $\psi \in l_n(y), \psi' \in l_n(z)$, *coherently* extend to $(\mathbb{F}_{n+1}, l_{n+1}) \supseteq (\mathbb{F}_n, l_n)$ so that $x \neq \sup_{n+1}\{y, z\}$.

– Similarly, for $\langle \sup \rangle \chi\chi' \in l_n(x)$.

(1): axiomatizing MIL_{Pre}

Axiomatization (soundness and completeness)

MIL_{Pre} is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \to \langle \text{sup} \rangle pq$

(4) $\langle P \rangle \langle P \rangle p \to \langle P \rangle p$

(Co.) $\langle \text{sup} \rangle pq \to \langle \text{sup} \rangle qp$

(Dk.) $(p \wedge \langle \text{sup} \rangle qr) \to \langle \text{sup} \rangle pq$

Proof idea

Soundness ✓

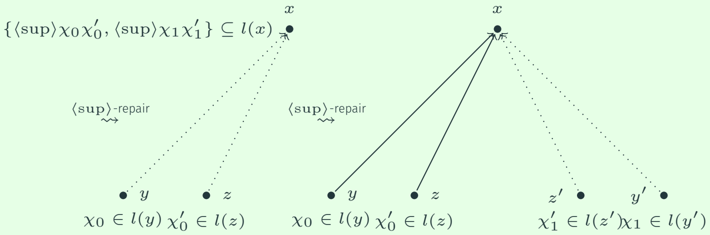
For completeness, let $\Gamma \supseteq \Gamma_0$ be an MCS extending some consistent Γ_0. We construct a satisfying model using the step-by-step method:

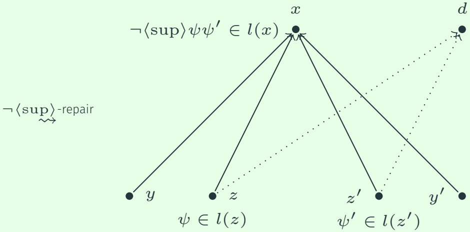
(Base) Singleton frame $\mathbb{F}_0 := (\{x_0\}, \{(x_0, x_0)\})$ and 'labeling' $l_0(x_0) = \Gamma$.

(Ind) Suppose $(\mathbb{F}_n, l_n)$ has been constructed.
– If $x \in \mathbb{F}_n$ and $\neg\langle \text{sup} \rangle \psi\psi' \in l_n(x)$ but $x = \text{sup}_n\{y, z\}$ s.t. $\psi \in l_n(y), \psi' \in l_n(z)$, *coherently* extend to $(\mathbb{F}_{n+1}, l_{n+1}) \supseteq (\mathbb{F}_n, l_n)$ so that $x \neq \text{sup}_{n+1}\{y, z\}$.
– Similarly, for $\langle \text{sup} \rangle \chi\chi' \in l_n(x)$.

Completeness of MIL_{Pre} (cont.)

Example





(1): axiomatizing MIL_{Pre}

Axiomatization (soundness and completeness)

MIL_{Pre} is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \rightarrow \langle \text{sup} \rangle pq$

(4) $\langle P \rangle \langle P \rangle p \rightarrow \langle P \rangle p$

(Co.) $\langle \text{sup} \rangle pq \rightarrow \langle \text{sup} \rangle qp$

(Dk.) $(p \wedge \langle \text{sup} \rangle qr) \rightarrow \langle \text{sup} \rangle pq$

About the proof

Soundness: routine.

Completeness: step-by-step method.

Corollary

As a corollary we get that $MIL_{Pre} = MIL_{Pos}$.

(1): axiomatizing MIL_{Pre}

Axiomatization (soundness and completeness)

MIL_{Pre} is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \to \langle \text{sup} \rangle pq$

(4) $\langle P \rangle \langle P \rangle p \to \langle P \rangle p$

(Co.) $\langle \text{sup} \rangle pq \to \langle \text{sup} \rangle qp$

(Dk.) $(p \wedge \langle \text{sup} \rangle qr) \to \langle \text{sup} \rangle pq$

About the proof

Soundness: routine.

Completeness: step-by-step method.

Corollary

As a corollary we get that $MIL_{Pre} = MIL_{Pos}$.

(2) and (3): ‘decidability via completeness’

(2) Find another class $\mathcal{C}$ for which $\mathrm{Log}(\mathcal{C}) = \mathit{MIL}_{\mathit{Pre}}$:

(i) Nothing in the ax. of $\mathit{MIL}_{\mathit{Pre}}$ necessitating ‘$\langle \mathrm{sup} \rangle$’ to be interpreted using a supremum relation.

(ii) Canon. re-interpretation:

$$\mathcal{C} := \{(W, C) \mid (W, C) \Vdash (Re.) \wedge (Co.) \wedge (4) \wedge (Dk.)\},$$

where $C \subseteq W^3$ is an arbitrary relation.

(iii) Then $\mathrm{Log}(\mathcal{C}) = \mathit{MIL}_{\mathit{Pre}}$.

(3) Decidability through FMP on $\mathcal{C}$:

(i) On $\mathcal{C}$, we get the FMP through filtration.

(ii) And this implies decidability.

Thus, we have solved both (A) and (D).

Gen. takeaway: *When dealing with ‘semantically introduced’ logics, not having the FMP (w.r.t. the class of definition) might not be very telling.*

(2) and (3): 'decidability via completeness'

(2) Find another class $\mathcal{C}$ for which $\mathrm{Log}(\mathcal{C}) = \mathit{MIL}_{\mathit{Pre}}$:

   (i) Nothing in the ax. of $\mathit{MIL}_{\mathit{Pre}}$ necessitating '$\langle \sup \rangle$' to be interpreted using a supremum relation.

   (ii) Canon. re-interpretation:

$$\mathcal{C} := \{(W, C) \mid (W, C) \Vdash (Re.) \wedge (Co.) \wedge (4) \wedge (Dk.)\},$$

   where $C \subseteq W^3$ is an arbitrary relation.

   (iii) Then $\mathrm{Log}(\mathcal{C}) = \mathit{MIL}_{\mathit{Pre}}$.

(3) Decidability through FMP on $\mathcal{C}$:

   (i) On $\mathcal{C}$, we get the FMP through filtration.

   (ii) And this implies decidability.

Thus, we have solved both (A) and (D).

Gen. takeaway: *When dealing with 'semantically introduced' logics, not having the FMP (w.r.t. the class of definition) might not be very telling.*

(2) and (3): 'decidability via completeness'

(2) Find another class $\mathcal{C}$ for which $\mathrm{Log}(\mathcal{C}) = \mathit{MIL}_{\mathit{Pre}}$:

   (i) Nothing in the ax. of $\mathit{MIL}_{\mathit{Pre}}$ necessitating '$\langle \mathrm{sup} \rangle$' to be interpreted using a supremum relation.

   (ii) Canon. re-interpretation:

$$\mathcal{C} := \{(W, C) \mid (W, C) \Vdash (Re.) \wedge (Co.) \wedge (4) \wedge (Dk.)\},$$

   where $C \subseteq W^3$ is an arbitrary relation.

   (iii) Then $\mathrm{Log}(\mathcal{C}) = \mathit{MIL}_{\mathit{Pre}}$.

(3) Decidability through FMP on $\mathcal{C}$:

   (i) On $\mathcal{C}$, we get the FMP through filtration.

   (ii) And this implies decidability.

Thus, we have solved both (A) and (D).

Gen. takeaway: *When dealing with 'semantically introduced' logics, not having the FMP (w.r.t. the class of definition) might not be very telling.*

(2) and (3): 'decidability via completeness'

(2) Find another class $\mathcal{C}$ for which $\mathrm{Log}(\mathcal{C}) = \mathit{MIL}_{\mathit{Pre}}$:

  (i) Nothing in the ax. of $\mathit{MIL}_{\mathit{Pre}}$ necessitating '$\langle \mathrm{sup} \rangle$' to be interpreted using a supremum relation.

  (ii) Canon. re-interpretation:

$$\mathcal{C} := \{(W, C) \mid (W, C) \Vdash (Re.) \wedge (Co.) \wedge (4) \wedge (Dk.)\},$$

  where $C \subseteq W^3$ is an arbitrary relation.

  (iii) Then $\mathrm{Log}(\mathcal{C}) = \mathit{MIL}_{\mathit{Pre}}$.

(3) Decidability through FMP on $\mathcal{C}$:

  (i) On $\mathcal{C}$, we get the FMP through filtration.

  (ii) And this implies decidability.

Thus, we have solved both (A) and (D).

Gen. takeaway: *When dealing with 'semantically introduced' logics, not having the FMP (w.r.t. the class of definition) might not be very telling.*

(2) and (3): ‘decidability via completeness’

(2) Find another class $\mathcal{C}$ for which $\mathrm{Log}(\mathcal{C}) = \mathit{MIL}_{\mathit{Pre}}$:

  (i) Nothing in the ax. of $\mathit{MIL}_{\mathit{Pre}}$ necessitating ‘$\langle \mathrm{sup} \rangle$’ to be interpreted using a supremum relation.

  (ii) Canon. re-interpretation:

$$\mathcal{C} := \{(W, C) \mid (W, C) \Vdash (Re.) \wedge (Co.) \wedge (4) \wedge (Dk.)\},$$

  where $C \subseteq W^3$ is an arbitrary relation.

  (iii) Then $\mathrm{Log}(\mathcal{C}) = \mathit{MIL}_{\mathit{Pre}}$.

(3) Decidability through FMP on $\mathcal{C}$:

  (i) On $\mathcal{C}$, we get the FMP through filtration.

  (ii) And this implies decidability.

Thus, we have solved both (A) and (D).

Gen. takeaway: *When dealing with ‘semantically introduced’ logics, not having the FMP (w.r.t. the class of definition) might not be very telling.*

(2) and (3): ‘decidability via completeness’

(2) Find another class $\mathcal{C}$ for which $\mathrm{Log}(\mathcal{C}) = \mathit{MIL}_{\mathit{Pre}}$:

   (i) Nothing in the ax. of $\mathit{MIL}_{\mathit{Pre}}$ necessitating ‘$\langle \mathrm{sup} \rangle$’ to be interpreted using a supremum relation.

   (ii) Canon. re-interpretation:

$$\mathcal{C} := \{(W, C) \mid (W, C) \Vdash (Re.) \wedge (Co.) \wedge (4) \wedge (Dk.)\},$$

   where $C \subseteq W^3$ is an arbitrary relation.

   (iii) Then $\mathrm{Log}(\mathcal{C}) = \mathit{MIL}_{\mathit{Pre}}$.

(3) Decidability through FMP on $\mathcal{C}$:

   (i) On $\mathcal{C}$, we get the FMP through filtration.

   (ii) And this implies decidability.

Thus, we have solved both (A) and (D).

Gen. takeaway: *When dealing with ‘semantically introduced’ logics, not having the FMP (w.r.t. the class of definition) might not be very telling.*

(2) and (3): ‘decidability via completeness’

(2) **Find another class $\mathcal{C}$ for which $\mathrm{Log}(\mathcal{C}) = \textit{MIL}_{\textit{Pre}}$:**

   (i) Nothing in the ax. of $\textit{MIL}_{\textit{Pre}}$ necessitating ‘$\langle \mathrm{sup} \rangle$’ to be interpreted using a supremum relation.

   (ii) Canon. re-interpretation:

$$\mathcal{C} := \{(W, C) \mid (W, C) \Vdash (Re.) \land (Co.) \land (4) \land (Dk.)\},$$

   where $C \subseteq W^3$ is an arbitrary relation.

   (iii) Then $\mathrm{Log}(\mathcal{C}) = \textit{MIL}_{\textit{Pre}}$.

(3) **Decidability through FMP on $\mathcal{C}$:**

   (i) On $\mathcal{C}$, we get the FMP through filtration.

   (ii) And this implies decidability.

Thus, we have solved both (A) and (D).

Gen. takeaway: *When dealing with ‘semantically introduced’ logics, not having the FMP (w.r.t. the class of definition) might not be very telling.*

(2) and (3): ‘decidability via completeness’

(2) **Find another class $\mathcal{C}$ for which $\mathrm{Log}(\mathcal{C}) = \mathit{MIL}_{\mathit{Pre}}$:**

   (i) Nothing in the ax. of $\mathit{MIL}_{\mathit{Pre}}$ necessitating ‘$\langle \mathrm{sup} \rangle$’ to be interpreted using a supremum relation.

   (ii) Canon. re-interpretation:

$$\mathcal{C} := \{(W, C) \mid (W, C) \Vdash (Re.) \wedge (Co.) \wedge (4) \wedge (Dk.)\},$$

   where $C \subseteq W^3$ is an arbitrary relation.

   (iii) Then $\mathrm{Log}(\mathcal{C}) = \mathit{MIL}_{\mathit{Pre}}$.

(3) **Decidability through FMP on $\mathcal{C}$:**

   (i) On $\mathcal{C}$, we get the FMP through filtration.

   (ii) And this implies decidability.

Thus, we have solved both (A) and (D).

Gen. takeaway: *When dealing with ‘semantically introduced’ logics, not having the FMP (w.r.t. the class of definition) might not be very telling.*

(2) and (3): ‘decidability via completeness’

(2) **Find another class $\mathcal{C}$ for which $\mathrm{Log}(\mathcal{C}) = \mathit{MIL_{Pre}}$:**

  (i) Nothing in the ax. of $\mathit{MIL_{Pre}}$ necessitating ‘$\langle \mathrm{sup} \rangle$’ to be interpreted using a supremum relation.

  (ii) Canon. re-interpretation:

$$\mathcal{C} := \{(W, C) \mid (W, C) \Vdash (Re.) \wedge (Co.) \wedge (4) \wedge (Dk.)\},$$

  where $C \subseteq W^3$ is an arbitrary relation.

  (iii) Then $\mathrm{Log}(\mathcal{C}) = \mathit{MIL_{Pre}}$.

(3) **Decidability through FMP on $\mathcal{C}$:**

  (i) On $\mathcal{C}$, we get the FMP through filtration.

  (ii) And this implies decidability.

Thus, we have solved both (A) and (D).

Gen. takeaway: *When dealing with ‘semantically introduced’ logics, not having the FMP (w.r.t. the class of definition) might not be very telling.*

How about join-semilattices (i.e., MIL_{Sem})?

Axiomatizing MIL_{Sem}

Three ways to completeness (some intuitions for our proof):

Henkin (e.g., $\mathbf{K}$)

Standard step-by-step (e.g., MIL_{Pre})

‘Indeterministic step-by-step’ (MIL_{Sem})

Axiomatizing MIL_{Sem}

Three ways to completeness (some intuitions for our proof):

Henkin (e.g., $\mathbb{K}$)

Standard step-by-step (e.g., MIL_{Pre})

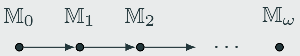
‘Indeterministic step-by-step’ (MIL_{Sem})

Axiomatizing MIL_{Sem}

Three ways to completeness (some intuitions for our proof):

Henkin (e.g., $\mathbf{K}$)

Standard step-by-step (e.g., MIL_{Pre})



'Indeterministic step-by-step' (MIL_{Sem})

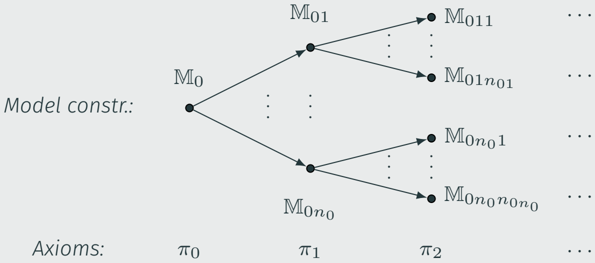
Axiomatizing MIL_{Sem}

Three ways to completeness (some intuitions for our proof):

Henkin (e.g., $\mathbf{K}$)

Standard step-by-step (e.g., MIL_{Pre})

'Indeterministic step-by-step' (MIL_{Sem})



Summary

What the thesis included:

- Surveyed the landscape of MILs on preorders and posets.[1]
- Made crossings with the Lambek Calculus and truthmaker semantics.[2]
- Axiomatized MIL_{Sem}.

What came next:

- MIL_{Sem} is not finitely axiomatizable.
- MIL_{Sem} is undecidable (and so are Hyperboolean algebras and more).
- Urquhart's relevance logic $\mathbf{S}$ is undecidable.

---

[1]See SBK (2023b).
[2]See SBK (2023a) (or thesis) for this, including proofs of FMP, decidability (and compactness) of a family of truthmaker logics.

References I

Knudstorp, S. B. (2023a). **“Logics of Truthmaker Semantics: Comparison, Compactness and Decidability”.** In: *Synthese* (cit. on p. 41).

— (2023b). **“Modal Information Logics: Axiomatizations and Decidability”.** In: *Journal of Philosophical Logic* (cit. on p. 41).

Van Benthem, J. (1996). **“Modal Logic as a Theory of Information”.** In: *Logic and Reality. Essays on the Legacy of Arthur Prior.* Ed. by J. Copeland. Clarendon Press, Oxford, pp. 135–168 (cit. on pp. 6–11).

— (10/2017). **“Constructive agents”.** In: *Indagationes Mathematicae* 29. DOI: `10.1016/j.indag.2017.10.004` (cit. on pp. 6–11).

References II

Van Benthem, J. (2019). "Implicit and Explicit Stances in Logic". In: *Journal of Philosophical Logic* 48.3, pp. 571–601. DOI: *10.1007/s10992-018-9485-y* (cit. on pp. 6–11).

Thank you!